# An Iterative Fitting Method for 2D Supercritical Steady Free Surface Flow

**Toon Demeester**[*], **E. Harald van Brummelen**[†], **Joris Degroote**[*,‡]
[*]Ghent University, Belgium
[†]Eindhoven University of Technology, the Netherlands
[‡]Flanders Make, Belgium
toon.demeester@ugent.be

## 1 Introduction

Computational methods to simulate incompressible, viscous water-air flows around ships can be divided in two categories: surface capturing and surface fitting methods. Capturing methods reconstruct the position of the free surface. An example is the volume-of-fluid method. In fitting methods the mesh lies along the free surface and deforms with it, so that its position is always known exactly. The air phase is often neglected, leading to one phase flow with free surface boundary conditions. The dynamic boundary condition (DBC) requires continuity of the stresses, but it is often assumed that the shear stresses are zero so that the DBC reduces to a condition of constant pressure at the free surface. The kinetic boundary condition (KBC) requires that the free surface is impermeable. For steady flows this means that the velocity must be parallel to the surface.

It is not possible to apply both these conditions to the free surface simultaneously. For this reason fitting methods are iterative techniques consisting of two steps: first the flow field is calculated with a fixed free surface position and suitable boundary conditions, then the free surface position is updated. These steps are repeated until the KBC and DBC are met to the required precision. For steady free surface flows, two clearly different fitting methodologies can be found in literature. The first one uses the DBC in the first step and the KBC in the second (Tzabiras (1997); Muzaferija and Perić (1997)). Using the KBC for updating the surface results in a time-stepping method, which is not efficient for steady cases due to the large number of time steps before transient phenomena disappear. The second methodology uses a combined boundary condition (KBC + DBC) in the flow solver and the DBC for the surface update. This *steady iterative* method by van Brummelen et al. (2001) is efficient but requires a dedicated coupled flow solver, making it less flexible.

This paper proposes a 2D fitting method which combines a steady iterative approach with a black-box flow solver. This requires that the boundary conditions in the flow solver are easy to implement (only KBC) and that the update procedure is time-independent (only DBC). Section 2 describes the theory, Section 3 the numerical results and Section 4 the conclusion.

## 2 Theory

Some notations are first introduced. Arrays are distinguished from scalars by the use of bold symbols ($\boldsymbol{z}$ versus $z$). A superscript (usually $k$) is used to denote the iteration index of the fitting method. Differences between values from consecutive iterations are written as $\Delta\boldsymbol{z}^k = \boldsymbol{z}^{k+1} - \boldsymbol{z}^k$.

### 2.1 Perturbation analysis

Before deriving the new fitting method, a basic free surface flow is investigated to better comprehend the nature of the problem at hand. The solution of an inviscid flow over a horizontal plate (see Fig. 1) is a flat free surface: for any depth $h$, the free surface pressure $\boldsymbol{p}$ will be constant (DBC) and the streamlines parallel to the surface (KBC). If the height $\boldsymbol{y}$ of the surface is perturbed with an arbitrary $\Delta\boldsymbol{y}$, the pressure will change with a certain $\Delta\boldsymbol{p}$. The relation between these two was derived by Demeester et al. (2017) for small perturbations. The conclusions are summarized here.

The free surface height is perturbed with

$$\Delta\boldsymbol{y} = a \sin(k\boldsymbol{x} + \theta) \tag{1}$$

where $k = 2\pi/\lambda$ is the wavenumber, $\theta$ an arbitrary phase angle and the amplitude $a$ small relative to $\lambda$

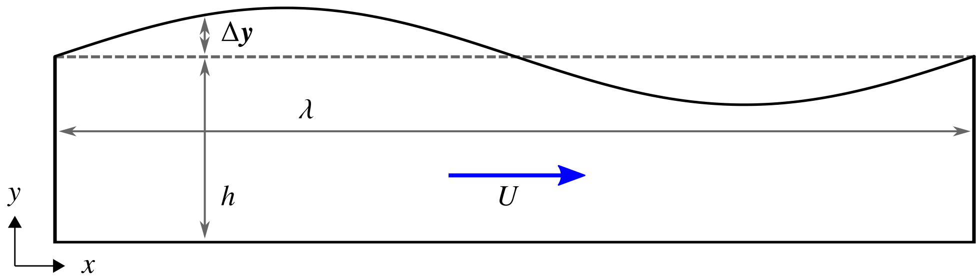


Fig. 1: Flow over a horizontal plate with perturbation $\Delta \boldsymbol{y}$.

and $h$. The pressure perturbation $\Delta \boldsymbol{p}$ then follows from a proportional relation:

$$\Delta \boldsymbol{p} = L\,\Delta \boldsymbol{y} \qquad \text{with} \qquad L = \rho g \left( \mathrm{Fr}^2 \frac{kh}{\tanh kh} - 1 \right). \tag{2}$$

$\rho$ is the fluid density, $g$ the gravitational acceleration and $\mathrm{Fr} = U/\sqrt{gh}$ the Froude number based on the depth $h$ and the average flow velocity $U$. Eq. (2) is graphically shown in Fig. 2 as a function of the dimensionless groups $kh$ and Fr. For clarity $1/L$ is plotted instead of $L$. An asymptote splits the domain in a region where gravitational forces dominate the flow (lower left corner) and a region where inertial forces dominate. At the asymptote (where $1/L \to \pm\infty$ or $L \to 0$), the opposing effects of gravity and inertia on $\Delta \boldsymbol{p}$ balance each outer out. If $L$ is set to zero in Eq. (2), the dispersion relation for gravitational waves is found (neglecting surface tension, Johnson (1997)). As a consequence, waves with $L = 0$ have their phase velocity equal to the average flow velocity $U$, so that they appear stationary. These so-called *steady gravity waves* are solutions of the linearized free surface problem and have $\Delta \boldsymbol{p} = 0$ for a non-zero $\Delta \boldsymbol{y}$. $L$ can be zero only if the flow is subcritical ($\mathrm{Fr} < 1$). In that case no unique $\boldsymbol{y}$ exists for a given $\boldsymbol{p}$. As this complicates the construction of a free surface fitting method, only supercritical flow ($\mathrm{Fr} > 1$) is treated in this paper.

### 2.2 Surrogate model

The free surface flow over a horizontal plate as described in Section 2.1 can also be simulated with a CFD solver, returning the free surface pressure perturbation $\Delta \boldsymbol{p}$ for an arbitrary height perturbation $\Delta \boldsymbol{y}$. In this section, a surrogate model will be constructed which approximates the behavior of this CFD solver.

Consider a numerical simulation where the free surface is discretized in the x-direction with $n$ equidistant points, with the y-values collected in an array $\boldsymbol{y} \in \mathbb{R}^{n,1}$. It is assumed that these points are ordered from inlet to outlet, i.e. $\boldsymbol{y}(0)$ and $\boldsymbol{y}(n-1)$ are respectively the free surface heights at inlet and outlet of the domain. The height $\boldsymbol{y}$ is now perturbed with an arbitrary (not necessarily small) $\Delta \boldsymbol{y}$, which can be decomposed into $n$ Fourier modes: a constant mode, `floor(n/2)` cosine modes and `floor((n-1)/2)` sine modes. It is assumed that for each[1] mode Eq. (2) is approximately valid, so that the corresponding $\Delta \boldsymbol{p} \in \mathbb{R}^{n,1}$ can be estimated. This operation can be written in matrix form as

$$\Delta \boldsymbol{p} = \boldsymbol{F}\,\Delta \boldsymbol{y} = \sum_{j=0}^{n-1} L_j\,\boldsymbol{\Phi}_j\,\Delta \boldsymbol{y} \qquad \text{with} \qquad \boldsymbol{\Phi}_j = \phi_j\,\phi_j^T \tag{3}$$

where $\phi_j$ is the normalized basis vector for Fourier mode $j$ and $L_j$ the corresponding factor from Eq. (2).

If there is a difference between inlet and outlet height, erroneous high frequency components appear in the Fourier decomposition. The solution is to subtract a sawtooth wave such that inlet and outlet heights become equal. The sawtooth is considered to have frequency zero (it is multiplied with $L_0$), instead of being decomposed into a spectrum of low and high frequencies. The sawtooth is identified by

[1]For the constant mode it is assumed that $\lambda \to \infty$ or $k \to 0$ so that $L_0 = \lim_{k \to 0} L = \rho g \left(\mathrm{Fr}^2 - 1\right)$.

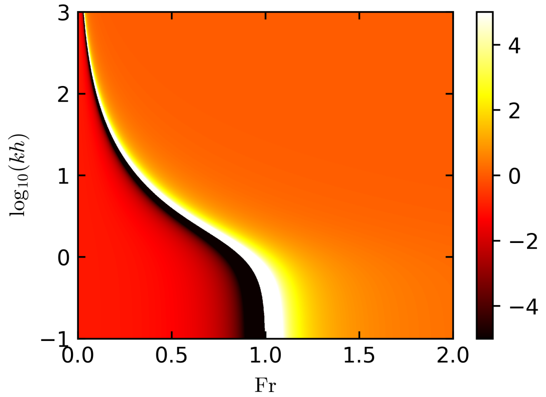


Fig. 2: $1/L$ as a function of Fr and $kh$.

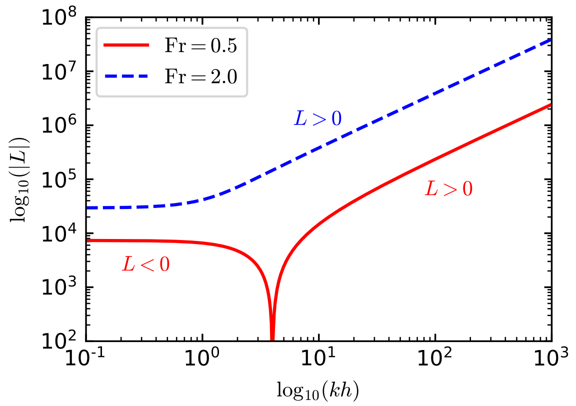


Fig. 3: $|L|$ as a function of $kh$ for Fr = 0.5 (subcritical) and Fr = 2.0 (supercritical).

multiplying $\Delta \boldsymbol{y}$ with the matrix

$$\boldsymbol{\Phi}_t = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{2} - \frac{i}{n-1} & 0 & & 0 & \frac{i}{n-1} - \frac{1}{2} \\ \vdots & \vdots & & \vdots & \vdots \\ -\frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix} \tag{4}$$

where $\boldsymbol{\Phi}_t \in \mathbb{R}^{n,n}$ and $i$ is the row index. Eq. (3) is accordingly adapted to

$$\Delta \boldsymbol{p} = \boldsymbol{F}_t \, \Delta \boldsymbol{y} \qquad \text{with} \qquad \boldsymbol{F}_t = L_0 \, \boldsymbol{\Phi}_t + \boldsymbol{F} \, (\boldsymbol{I}_n - \boldsymbol{\Phi}_t) \, . \tag{5}$$

The matrix $\boldsymbol{F}_t$ serves as a surrogate model for simulations of flow over a horizontal plate. Assuming perturbations of any 2D free surface flow behave similarly to those of flow over a horizontal plate, $\boldsymbol{F}_t$ can be used as a surrogate model for a variety of cases, although it will likely be less accurate then.

### 2.3 Free surface fitting method

The free surface problem is now stated more precisely, using the same free surface discretization as in the previous section ($\boldsymbol{y}, \boldsymbol{p} \in \mathbb{R}^{n,1}$ and uniformly distributed):

> given a black-box flow solver $\mathcal{G}(\boldsymbol{y}) = \boldsymbol{p}$ which fulfills the KBC,
> find $\boldsymbol{y}$ so that $\boldsymbol{p} = p_{cst}\mathbf{1}$ and $\boldsymbol{y}(0) = y_0$.

The KBC is fulfilled by using a free-slip wall in the flow solver. It is assumed that the function $\mathcal{G}$ is a bijection[2], i.e. with each $\boldsymbol{y}$ corresponds a unique $\boldsymbol{p}$ and vice-versa. As $\boldsymbol{p}$ can be equal to an arbitrary constant $p_{cst}$ ($\mathbf{1}$ is the all-ones array), a condition on the inlet $\boldsymbol{y}(0)$ has to be added to close the system and assure uniqueness of the solution $\boldsymbol{y}$. The value of $p_{cst}$ is not of interest and can be removed from the problem by filtering out the constant mode (average value). With $\boldsymbol{p}_\emptyset = (\boldsymbol{I}_n - \boldsymbol{\Phi}_0)\,\boldsymbol{p}$ and $\mathcal{F} = (\boldsymbol{I}_n - \boldsymbol{\Phi}_0)\,\mathcal{G}$ the free surface problem becomes:

> given a black-box flow solver $\mathcal{F}(\boldsymbol{y}) = \boldsymbol{p}_\emptyset$ which fulfills the KBC,
> find $\boldsymbol{y}$ so that $\boldsymbol{p}_\emptyset = 0$ and $\boldsymbol{y}(0) = y_0$.

This problem has to be solved iteratively: in each iteration $k$ the flowfield is solved by calling $\mathcal{F}$, then the new position $\boldsymbol{y}^{k+1} = \boldsymbol{y}^k + \Delta\boldsymbol{y}^k$ is calculated and the mesh updated accordingly. Newton's method can be used to calculated $\Delta\boldsymbol{y}^k$ as

$$\mathcal{F}' \, \Delta\boldsymbol{y}^k = -\boldsymbol{p}_\emptyset^k \tag{6}$$

[2] There is no strict proof that the non-linear black-box $\mathcal{G}$ is bijective, but the linear model for an inviscid flow clearly is for Fr > 1 (see Eq. 3 and Fig. 3)

with $\mathcal{F}'$ the Jacobian of $\mathcal{F}$. As the flow solver is a black-box, its Jacobian is not known, but an approximation $\widehat{\mathcal{F}'}$ can be used. Rewriting the inlet condition, a linear system emerges to calculate $\Delta \boldsymbol{y}^k$:

$$\begin{cases} \widehat{\mathcal{F}'} \Delta \boldsymbol{y}^k = -\boldsymbol{p}_\emptyset^k \\ \Delta \boldsymbol{y}^k(0) = y_0 - \boldsymbol{y}^k(0) \end{cases} \tag{7}$$

The surrogate model $\boldsymbol{F}_t$ from Eq. (5) is used in a first approximation to $\mathcal{F}'$:

$$\widehat{\mathcal{F}'_{pert}} = (\boldsymbol{I}_n - \Phi_0)\boldsymbol{F}_t \tag{8}$$

To stabilize and accelerate the iterations, a second approximation for $\mathcal{F}'$ is constructed using the IQN-ILS algorithm by Degroote et al. (2009). This is used in fluid-structure-interaction to improve convergence of coupling iterations. The idea is to use known input-output pairs of a black-box system to construct a better approximation of its Jacobian than is available at that moment. In this case, the input of the black-box $\mathcal{F}$ is $\boldsymbol{y}$, the output $\boldsymbol{p}_\emptyset$. Differences between consecutive iterations are stored in the matrices

$$\boldsymbol{V}^k = \begin{bmatrix} \Delta \boldsymbol{y}^{k-1} & \cdots & \Delta \boldsymbol{y}^0 \end{bmatrix}, \tag{9}$$

$$\boldsymbol{W}^k = \begin{bmatrix} \Delta \boldsymbol{p}_\emptyset^{k-1} & \cdots & \Delta \boldsymbol{p}_\emptyset^0 \end{bmatrix} \tag{10}$$

from which an approximate Jacobian is constructed as

$$\widehat{\mathcal{F}'_{IQN}}^k = \boldsymbol{W}^k \boldsymbol{R}^{k^{-1}} \boldsymbol{Q}^{k^T} \qquad \text{with} \qquad \boldsymbol{V}^k = \boldsymbol{Q}^k \boldsymbol{R}^k \quad \text{(economy-size QR-decomposition).} \tag{11}$$

There are now two approximations of the Jacobian, which have to be combined. The Jacobian based on the perturbation analysis (Eq. (8)) is constant, the one based on IQN-ILS (Eq. (11)) changes in each iteration as the matrices $\boldsymbol{V}^k$ and $\boldsymbol{W}^k$ grow bigger. $\widehat{\mathcal{F}'_{IQN}}$ only affects the part of $\Delta \boldsymbol{y} \in \text{span}(\boldsymbol{V})$, which is extracted as $\boldsymbol{Q}^k \boldsymbol{Q}^{k^T} \Delta \boldsymbol{y}$. $\widehat{\mathcal{F}'_{pert}}$ is used for the remaining part of $\Delta \boldsymbol{y}$.

The linear system to calculate $\Delta \boldsymbol{y}^k$ is

$$\begin{cases} \widehat{\mathcal{F}'}^k \Delta \boldsymbol{y}^k = -\boldsymbol{p}_\emptyset^k \\ \Delta \boldsymbol{y}^k(0) = y_0 - \boldsymbol{y}^k(0) \end{cases} \tag{12}$$

with

$$\widehat{\mathcal{F}'}^k = \widehat{\mathcal{F}'_{IQN}}^k \boldsymbol{Q}^k \boldsymbol{Q}^{k^T} + \widehat{\mathcal{F}'_{pert}} \left( \boldsymbol{I}_n - \boldsymbol{Q}^k \boldsymbol{Q}^{k^T} \right) \tag{13}$$

$$= \boldsymbol{W}^k \boldsymbol{R}^{k^{-1}} \boldsymbol{Q}^{k^T} + (\boldsymbol{I}_n - \boldsymbol{\Phi_0})\, \boldsymbol{F}_t \left( \boldsymbol{I}_n - \boldsymbol{Q}^k \boldsymbol{Q}^{k^T} \right). \tag{14}$$

The Jacobian approximation $\widehat{\mathcal{F}'}$ has rank $n - 1$, because the constant mode is not part of its column space. By adding the inlet condition, the system in Eq. (12) with $n$ unknowns becomes rank $n$ and has a unique solution. However, as it has $n + 1$ equations for $n$ unknowns it must be solved with a least-squares method.

Algorithm 1 outlines the fitting method. For compactness the left-hand side matrix and right-hand side array of Eq. (12) are denoted respectively by $\boldsymbol{A}^k \in \mathbb{R}^{n+1,n}$ and $\boldsymbol{b}^k \in \mathbb{R}^{n+1,1}$, so that the system can be written in the form $\boldsymbol{A}^k \Delta \boldsymbol{y}^k = \boldsymbol{b}^k$.

## 3 Numerical experiments

The new method is tested on the academic test case of 2D flow over an obstacle as shown in Fig. 4 (flow from left to right). The experimental data was acquired by Cahouet (1984) and has been used in several papers for evaluating surface fitting methods (Tzabiras (1997); van Brummelen et al. (2001)). The shape of the obstacle is described by

$$y_{obst}(x) = \frac{27}{4} \frac{H}{L^3} x(x - L)^2 \qquad \text{for} \qquad 0 \leq x \leq L \tag{15}$$

**Algorithm 1** Free surface fitting method.

$k = 0$

$\boldsymbol{p}_\emptyset^0 = \mathcal{F}\left(\boldsymbol{y}^0\right)$ $\qquad \triangleright\ \mathcal{F} = (\boldsymbol{I}_n - \boldsymbol{\Phi}_0)\,\mathcal{G}$

**while** $\left\|\boldsymbol{p}_\emptyset^k\right\|_2 > \varepsilon$ **do**

  **if** $k > 0$ **then**

    construct $\boldsymbol{V}^k$, $\boldsymbol{W}^k$

    calculate QR-decomposition $\boldsymbol{V}^k = \boldsymbol{Q}^k\boldsymbol{R}^k$

  **end if**

  construct $\boldsymbol{A}^k$ and $\boldsymbol{b}^k$ $\qquad \triangleright\ \boldsymbol{A}^k$ and $\boldsymbol{b}^k$ from Eq. (12)

  solve $\boldsymbol{A}^k\Delta\boldsymbol{y}^k = \boldsymbol{b}^k$ with least-squares

  $\boldsymbol{y}^{k+1} = \boldsymbol{y}^k + \Delta\boldsymbol{y}^k$

  $k = k + 1$

  $\boldsymbol{p}_\emptyset^k = \mathcal{F}\left(\boldsymbol{y}^k\right)$

**end while**

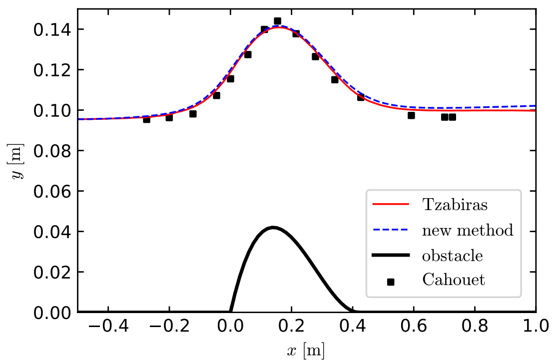


Fig. 4: Free surface height compared to experimental data from Cahouet (1984) and numerical data from Tzabiras (1997).

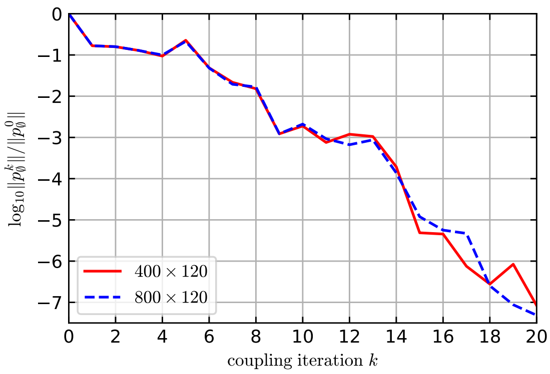


Fig. 5: Relative norm of free surface pressure $\boldsymbol{p}_\emptyset^k$ for two meshes.

where $H = 0.042$m is the height and $L = 0.42$m the length of the obstacle. The inlet water depth is 0.09545m, the Froude number is 2.05. From these two, the inlet velocity can be determined. Boundary conditions are a uniform inlet velocity, a hydrostatic pressure outlet, a free-slip wall at the free surface and a no-slip wall at the bottom. A structured mesh of $400 \times 120$ cells ($x \times y$) is used. Exponential growth is used at the bottom to resolve the boundary layer, giving $1 < y^+ < 5$. The standard $k - \epsilon$ model is used for turbulence modeling. To assess convergence of the fitting method, the pressure $\boldsymbol{p}_\emptyset$ at the free surface is monitored. The initial height $\boldsymbol{y}^0$ is a horizontal free surface.

Fig. 4 compares the new method to the experimental results by Cahouet (1984) and to the simulations by Tzabiras (1997), all results agree well. Fig. 5 shows the decrease of the free surface pressure $\boldsymbol{p}_\emptyset$ during the coupling iterations for the original mesh and for a mesh which is refined in the x-direction. Both converge 7 orders of magnitude in 20 iterations. No influence of the number of Fourier modes (i.e. the mesh size) can be seen.

The new method cannot easily be compared to the time-stepping method of Tzabiras (1997) in terms of speed due to a lack of data. The number of coupling iterations (i.e. surface updates) for a certain decrease in residual can be compared, but it must be kept in mind that the residual's definition is very different[3]. For a similar setup, the time-stepping method needed ~500 coupling iterations for the residual to decrease by one order of magnitude, compared to 20 iterations for 7 orders with the new method.

[3]In Tzabiras (1997), the DBC is used in the flow solver, so the residual is based on the KBC, i.e. the mass flow rate through the free surface.

## 4 Conclusion

A novel fitting method is introduced for 2D supercritical steady free surface flows, based on a perturbation analysis of the free surface. The method successfully combines a steady iterative approach with a black-box flow solver. The IQN-ILS technique is added to improve stability and accelerate convergence.

The method is compared to both experimental and numerical reference data and shows good correspondence. Convergence is reached in a small number of iterations and seems to be independent of mesh size. Currently the method is being extended to subcritical flows, after which it will be generalized to 3D.

## References


Cahouet, J. (1984). *Etude numérique et expérimentale du problème bidimensionnel de la résistance de vagues non-linéaire*. Ecole nationale supérieure de techniques avançées.

Degroote, J., Bathe, K.-J., and Vierendeels, J. (2009). Performance of a new partitioned procedure versus a monolithic procedure in fluid–structure interaction. *Computers & Structures*, 87(11):793–801.

Demeester, T., Degroote, J., and Vierendeels, J. (2017). Stability analysis of a partitioned iterative method for steady free surface flow. *Journal of Computational Physics*.

Johnson, R. S. (1997). *A modern introduction to the mathematical theory of water waves*, volume 19. Cambridge University Press.

Muzaferija, S. and Perić, M. (1997). Computation of free-surface flows using the finite-volume method and moving grids. *Numerical Heat Transfer*, 32(4):369–384.

Tzabiras, G. (1997). A numerical investigation of 2D, steady free surface flows. *International Journal for Numerical Methods in Fluids*, 25(5):567–598.

van Brummelen, E., Raven, H., and Koren, B. (2001). Efficient numerical solution of steady free-surface Navier–Stokes flow. *Journal of Computational Physics*, 174(1):120–137.